PAPER

An exact formula for electromagnetic momentum in terms of the charge density and the Coulomb gauge vector potential

IOP Publishing European Journal of Physics
Eur. J. Phys. **39** (2018) 025202 (9pp) https://doi.org/10.1088/1361-6404/aa9051

An exact formula for electromagnetic momentum in terms of the charge density and the Coulomb gauge vector potential

Hanno Essén

Department of Mechanics Royal Institute of Technology SE-100 44 Stockholm, Sweden

E-mail: hanno@mech.kth.se

Received 4 September 2017, revised 24 September 2017
Accepted for publication 2 October 2017
Published 9 January 2018

Abstract

The electromagnetic momentum $\boldsymbol{p} = 1/(4\pi c) \int \boldsymbol{E} \times \boldsymbol{B} \mathrm{d}V$ is sometimes approximated by $\boldsymbol{p}_0 = (1/c) \int \varrho \boldsymbol{A} \mathrm{d}V$, where ϱ is the charge density and $\boldsymbol{A}$ is the Coulomb gauge vector potential. Here, we show that $\boldsymbol{p}_0$ is the first term in an exact two-term expression $\boldsymbol{p} = \boldsymbol{p}_0 + \boldsymbol{p}_1$ where the second term refers to radiation. When the charge density is zero, $\boldsymbol{p} = \boldsymbol{p}_1$ is the momentum of fields propagating in vacuum. In the presence of charged particles, however, $\boldsymbol{p}_0$ normally dominates. We argue that $\boldsymbol{p}_0$ is the natural formula for the electromagnetic momentum when radiation can be neglected. It is shown that this term may in fact be much larger than the purely mechanical contribution from mass times velocity.

Keywords: electromagnetic momentum, Coulomb gauge, Darwin Lagrangian, canonical momentum

1. Introduction

Electromagnetic momentum is the contribution to the momentum of a system from electromagnetic fields which must be added to the usual momentum, due to mass times velocity, to the get the total momentum that is conserved for a closed isolated system. Fundamental considerations involving the energy momentum tensor of the electromagnetic field give the electromagnetic momentum density (in Gaussian units) as

$$\boldsymbol{G} = \frac{1}{4\pi c}\boldsymbol{E} \times \boldsymbol{B}. \tag{1}$$

The momentum is then the volume integral of this density

$$\boldsymbol{p} = \frac{1}{4\pi c}\int \boldsymbol{E} \times \boldsymbol{B}\, \mathrm{d}V. \tag{2}$$

The momentum density vector field is proportional to the Poynting vector $\boldsymbol{S} = c^2\boldsymbol{G}$, giving the energy flux density in the field.

Here, we will analyze this expression using concepts of classical electrodynamics. We will find that this this expression, in a natural way, can be divided into two parts $\boldsymbol{p} = \boldsymbol{p}_0 + \boldsymbol{p}_1$ of very different physical significance. One part, here denoted $\boldsymbol{p}_0$, which often is quite large, is due the presence of moving charged particles and applies to electric currents in conductors and electric oscillations in plasmas—see equation (3), below. The remaining part, $\boldsymbol{p}_1$—normally much smaller—is due to electromagnetic waves propagating in vacuum. It is important to be aware of the existence of these two parts, and the distinction between them.

An approximate result, which goes all the way back to Maxwell, is that

$$\boldsymbol{p} \approx \boldsymbol{p}_0 = \frac{1}{c}\int \varrho \boldsymbol{A}\, \mathrm{d}V, \tag{3}$$

where ϱ is the charge density and $\boldsymbol{A}$ is the Coulomb gauge vector potential. This formula gives the otherwise somewhat abstract vector potential a concrete meaning, and is therefore of great interest. The circumstances under which this approximation is valid are often vaguely described as the quasi-static approximation, 'when higher order terms in v/c can be neglected' or 'when the effects of radiation and retardation can be ignored'. Discussions of this and other approximations are by Furry [1], Carpenter [2], Gsponer [3], Griffiths [4], McDonald [5], Lorrain [6], and Hu [7]. Interpretation of the vector potential as potential momentum per charge have been advocated by Calkin [8, 9], Konopinski [10, 11], Gingras [12], and Semon and Taylor [13], among others. These concepts have also been much discussed in connection with hidden momentum ideas (see e.g. Aguirregabiria *et al* [14, 15], Hnizdo [16], Johnson *et al* [17], Babson *et al* [18], Boyer [19], Redfern [20]).

After deriving $\boldsymbol{p} = \boldsymbol{p}_0 + \boldsymbol{p}_1$, and discussing the nature of the two terms, we focus on $\boldsymbol{p}_0$, and motivate it as the natural momentum expression arising from the Darwin Lagrangian [21]. We also estimate the electromagnetic contribution to the total momentum as compared to the kinetic contribution $\boldsymbol{MV}$ for a system of collectively moving charged particles. For macroscopic numbers of particles, the electromagnetic contribution in fact dominates—a fact rarely pointed out in the literature. From all this, we conclude that (3) may be regarded as the most natural and useful expression for electromagnetic momentum, as long as radiation can be neglected.

2. The two parts of the electromagnetic momentum density

Maxwell's homogeneous equations are identically satisfied if one assumes that the fields are given by

$$\boldsymbol{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}, \quad \text{and} \quad \boldsymbol{B} = \nabla \times \boldsymbol{A}, \tag{4}$$

in terms of scalar ϕ and vector $\boldsymbol{A}$ potentials. When this is inserted into Maxwell's inhomogeneous equations, one finds (see Jackson [22], sections 6.2 and 6.3)

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{A}) = -4\pi\varrho, \tag{5}$$

$$\Box\boldsymbol{A} - \nabla\left(\frac{1}{c}\frac{\partial\phi}{\partial t} + \nabla \cdot \boldsymbol{A}\right) = -\frac{4\pi}{c}\boldsymbol{j}. \tag{6}$$

When the bracket in equation (6) is put to zero, the Lorenz gauge, both these equations become inhomogeneous wave equations. If one instead chooses the Coulomb (or transverse, or radiation) gauge by putting[1]

$$\nabla \cdot \boldsymbol{A} = 0, \tag{7}$$

one finds that

$$\nabla^2\phi = -4\pi\varrho, \tag{8}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\boldsymbol{A} = -\frac{4\pi}{c}\boldsymbol{j}_t. \tag{9}$$

Here,

$$\boldsymbol{j}_t = \boldsymbol{j} - \frac{1}{4\pi}\nabla\frac{\partial\phi}{\partial t} \tag{10}$$

is the transverse current density which thus acts as source of the Coulomb gauge vector potential. This current density is divergence free, $\nabla \cdot \boldsymbol{j}_t = 0$, as is seen using the equation of continuity.

To get the momentum density, we now insert equations (4) in the definition (1), to get

$$\boldsymbol{G} = \boldsymbol{G}_0 + \boldsymbol{G}_1, \tag{11}$$

where

$$\boldsymbol{G}_0 = -\frac{1}{4\pi c}\nabla\phi \times (\nabla \times \boldsymbol{A}), \tag{12}$$

$$\boldsymbol{G}_1 = -\frac{1}{4\pi c^2}\frac{\partial\boldsymbol{A}}{\partial t} \times (\nabla \times \boldsymbol{A}). \tag{13}$$

It is now elementary to show that

$$\boldsymbol{p}_0 = \int \boldsymbol{G}_0 \,\mathrm{d}V = \frac{1}{c}\int \varrho\boldsymbol{A}\,\mathrm{d}V, \tag{14}$$

assuming (7), (8), and that certain surface integrals go to zero at infinity. The derivation is given in an appendix. We see that the momentum is given entirely in terms of the charge density and the Coulomb gauge vector potential and its derivatives.

The exact electromagnetic momentum is $\boldsymbol{p} = \boldsymbol{p}_0 + \boldsymbol{p}_1$, where

$$\boldsymbol{p}_1 = \int \boldsymbol{G}_1 \,\mathrm{d}V = \frac{-1}{4\pi c^2}\int \frac{\partial\boldsymbol{A}}{\partial t} \times (\nabla \times \boldsymbol{A})\,\mathrm{d}V = \frac{1}{4\pi c^2}\int \boldsymbol{B} \times \frac{\partial\boldsymbol{A}}{\partial t}\,\mathrm{d}V. \tag{15}$$

This part is of higher order in v/c than $\boldsymbol{p}_0$, as shown below, and is related to radiation.

2.1. Orders of magnitude and dimensional analysis

In the Gaussian unit system, $\boldsymbol{A}$ has the same dimension as ϕ, namely charge Q divided by length L, but the vector potential is multiplied by v/c, where v is the speed of the charged

[1] This gauge condition, together with suitable boundary conditions, implies unique solutions.

particles producing the current—see (18) below. Also, Q^2/L has dimension energy, and therefore this quantity divided by c^2 has dimension mass M.

Using this, the dimension of $\boldsymbol{p}_0$ can be written

$$[\boldsymbol{p}_0] = \frac{1}{c^2}\frac{Q^2}{L}v = Mv. \tag{16}$$

Doing the same analysis for $\boldsymbol{p}_1$ (putting $[c\partial t] = L$ and $[\nabla] = 1/L$) gives

$$[\boldsymbol{p}_1] = \frac{1}{c^2}\frac{Q^2}{L}\frac{v}{c}v = \frac{v}{c}Mv. \tag{17}$$

We see that both quantities have dimension mass times velocity, as they should, but $\boldsymbol{p}_1$ is multiplied by an extra dimensionless factor v/c.

3. The message of the Darwin Lagrangian

The Darwin Lagrangian [21] describes most electromagnetic phenomena quite well, with the exception of radiation, which is neglected. The theory behind this Lagrangian is presented in well known textbooks such as Landau and Lifshitz [23, section 65] and Jackson [22, section 12.6]. More extensive discussions can be found in Page and Adams [24, section 96], Podolsky and Kunz [25, section 27], or Schwinger *et al* [26, equation (33.23)]. Basic articles of interest are Breitenberger [27], Kennedy [28], Essén [29, 30]. Various applications of the Darwin Lagrangian illustrating its usefulness can be found in Stettner [31], Boyer [32, 33], Essén [34–37], and Essén and Fiolhais [38].

The vector potential is not always mentioned in connection with the Darwin Lagrangian, but basically the central idea is to approximate the Liénard–Wiechert potentials. Landau and Lifshitz [23, section 65], make a gauge transformation to the Coulomb gauge after truncating series expansions of these. Jackson [22, section 12.6], argues that it is sufficient to simply solve the vector Poisson equation obtained by neglecting the time derivatives in equation (9). For a point charge e with velocity $\boldsymbol{v}$ at the origin, the resulting divergence free (Coulomb gauge) vector potential at $\boldsymbol{r}$ is given by

$$\boldsymbol{A}(\boldsymbol{r}, \boldsymbol{v}) = \frac{e}{2c}\frac{\boldsymbol{v} + (\boldsymbol{v}\cdot\hat{\boldsymbol{e}})\hat{\boldsymbol{e}}}{r}. \tag{18}$$

Here $\hat{\boldsymbol{e}}$ is the unit vector $\boldsymbol{r}/r$.

The Darwin Lagrangian can be written

$$L_{\mathrm{D}} = \sum_{a=1}^{N}\left[L_m(\boldsymbol{v}_a) - \frac{e_a}{2}\phi_a(\boldsymbol{r}_a) + \frac{e_a}{2c}\boldsymbol{v}_a\cdot\boldsymbol{A}_a(\boldsymbol{r}_a)\right], \tag{19}$$

where

$$\phi_a(\boldsymbol{r}) = \sum_{b(\neq a)}^{N}\frac{e_b}{|\boldsymbol{r} - \boldsymbol{r}_b|}, \tag{20}$$

and

$$\boldsymbol{A}_a(\boldsymbol{r}) = \sum_{b(\neq a)}^{N}\frac{e_b}{2c}\frac{[\boldsymbol{v}_b + (\boldsymbol{v}_b\cdot\hat{\boldsymbol{e}}_b)\hat{\boldsymbol{e}}_b]}{|\boldsymbol{r} - \boldsymbol{r}_b|}. \tag{21}$$

Here, $\hat{\boldsymbol{e}}_b = (\boldsymbol{r} - \boldsymbol{r}_b)/|\boldsymbol{r} - \boldsymbol{r}_b|$, while $L_m(\boldsymbol{v})$ is the Lagrangian of a free particle. Depending on the application this may or may not be relativistic. Normally the non-relativistic version $L_m(\boldsymbol{v}) = m\boldsymbol{v}^2/2$ will do.

For a Lagrangian L the canonical momentum of particle a is defined as

$$\boldsymbol{p}_a = \frac{\partial L}{\partial \dot{\boldsymbol{r}}_a} = \frac{\partial L}{\partial \boldsymbol{v}_a}. \tag{22}$$

and this applied to the Lagrangian (19) gives ($\hat{\boldsymbol{e}}_{ba} = (\boldsymbol{r}_a - \boldsymbol{r}_b)/|\boldsymbol{r}_a - \boldsymbol{r}_b|$)

$$\boldsymbol{p}_a = \boldsymbol{\pi}_a + \sum_{b(\neq a)} \frac{e_a e_b}{r_{ba}} \frac{[\boldsymbol{v}_b + \hat{\boldsymbol{e}}_{ba}(\boldsymbol{v}_b \cdot \hat{\boldsymbol{e}}_{ba})]}{2c^2}, \tag{23}$$

where $\boldsymbol{\pi}_a$ is the mechanical momentum, often simply $m_a\boldsymbol{v}_a$, but more generally $\partial L_{\mathrm{m}}/\partial \boldsymbol{v}_a$ which may include relativistic effects. This equation can be found in Schwinger *et al* [26] as equation (33.24). We can write this as

$$\boldsymbol{p}_a = \boldsymbol{\pi}_a + \frac{e_a}{c}\boldsymbol{A}_a(\boldsymbol{r}_a), \tag{24}$$

using equation (21).

If the charge density of point particles,

$$\varrho(\boldsymbol{r}) = \sum_a e_a \delta(\boldsymbol{r} - \boldsymbol{r}_a), \tag{25}$$

is inserted into equation (14) we find that

$$\boldsymbol{p}_0 = \sum_a \frac{e_a}{c}\boldsymbol{A}(\boldsymbol{r}_a). \tag{26}$$

This is evidently the sum of the electromagnetic parts of the canonical momenta of equation (24).

Hence the total electromagnetic momentum according to the Darwin Lagrangian is exactly given by the formula (14) when ϱ is the charge density of point particles. We conclude that the expression (14) and electromagnetic momentum as obtained from the Darwin Lagrangian agree. When one can neglect radiation, this should thus be considered as the most natural formula for the electromagnetic momentum.

3.1. The total momentum of collectively moving charges

Let us estimate the total momentum of N particles of mass m and charge e moving collectively with velocity $\boldsymbol{V}$. If we assume that the typical distance between the particles is R, we can get a rough estimate for the total momentum $\boldsymbol{P} = \sum_{a=1}^{N} \boldsymbol{p}_a$ using (23). We find

$$\boldsymbol{P} = Nm\boldsymbol{V} + N^2\frac{e^2}{c^2 R}\boldsymbol{V} = Nm\left(1 + \frac{e^2}{mc^2}\frac{N}{R}\right)\boldsymbol{V}. \tag{27}$$

If we put $M = Nm$ for the total (rest) mass, and note that $r_{\mathrm{e}} = \frac{e^2}{mc^2}$ is the classical electron radius, we find

$$\boldsymbol{P} = M\left(1 + \frac{Nr_{\mathrm{e}}}{R}\right)\boldsymbol{V} = M(1 + \nu)\boldsymbol{V}. \tag{28}$$

The dimensionless number $\nu = \frac{Nr_{\mathrm{e}}}{R}$, which might be called the Darwin number, is thus an indicator of whether the effective inertial mass of the moving charges is dominated by rest mass M ($\nu \ll 1$) or if it is dominated by the effective mass of the resulting magnetic energy ($\nu \gg 1$).

Darwin himself [39] studied the inertia of electrons in metals, and found that for ordinary conduction electrons the inductive inertia is many orders of magnitude larger than the sum of the rest masses of the electrons. Many textbooks derive the—so called—plasma frequency,

$$\omega_{\mathrm{p}} = \sqrt{\frac{ne^2}{m}} \tag{29}$$

where n is the number density of electrons. In that derivation, it is assumed that the inertia is due to the electron rest mass m. When the Darwin number is large, this is clearly not correct. Essén [34], in a calculation based on the Darwin Lagrangian shows that the mass in the denominator should be replaced by $m(1 + \nu)$ to get better estimates of the frequency of collective oscillations in plasmas.

4. Conclusions

The Coulomb gauge requirement, $\nabla \cdot \boldsymbol{A} = 0$, is crucial in the derivations above. The Darwin Lagrangian is, however, rarely written explicitly in the form (19) with the vector potential, even if the expression (18) is obtained as an intermediate step in Jackson [22], equation (12.80), as well as Landau and Lifshitz [23], equation (65.6). The choice of the Coulomb gauge has the result that relativistic effects, including *retardation*, are taken into account to order $(v/c)^2$ by the Darwin Lagrangian. This is somewhat obscured by Jackson's derivation, but is clear from Darwin's [21] original derivation, as well as in Landau's and Lifshitz'. A Coulomb gauge vector potential correct to all orders in v/c has been derived by Hnizdo [40], and its implications for the Darwin formalism have been discussed by Essén [41]. More information on the significance of the Coulomb gauge can be found in Jackson [42]. It is advantageous to use when there is a natural rest frame with respect to which velocities are small, so that relativistic invariance is of less importance.

One reason that the expression (14) for the electromagnetic momentum is rarely advocated is that the vector potential is not seen as unique, due to gauge freedom (see Berche *et al* [43]). However, as often pointed out the gauge freedom does not pose any real difficulty. The physically interesting vector potential should go to zero with distance from its source and should be singular at a point particle source, just as the Coulomb potential. One should also note that the Darwin vector potential (21) has no gauge freedom. It is given by a superposition of terms (18), one for each moving charged point particle, and this vector potential is the natural one to use.

Acknowledgments

The author is grateful to Drs L H Söderholm and A B Nordmark for valuable comments.

Appendix

The derivation of (14) given below is sketched in e.g. Trammell [44] and Kittel [45], see appendix G—but these derivations are a bit brief and hard to follow. A more careful proof is given below.

Using the three-dimensional antisymmetric unit tensor e_{ijk}, one can write the ith component of a vector product $(\boldsymbol{a} \times \boldsymbol{b})_i = e_{ijk} a_j b_k$ where summation over the repeated indices from 1 to 3 is implied. Using the Kronecker δ_{ij}, a scalar product can similarly be written $\boldsymbol{a} \cdot \boldsymbol{b} = \delta_{ij} a_i b_j = a_i b_i$. The ith component of $\boldsymbol{u} = \nabla\phi \times (\nabla \times \boldsymbol{A})$ can be written

$$u_i = e_{ijk} \partial_j \phi \; e_{klm} \partial_l A_m. \tag{30}$$

Here $\partial_i = \partial/\partial x_i$. From this, we get

$$u_i = e_{ijk} e_{klm} \partial_j \phi \; \partial_l A_m = e_{ijk} e_{lmk} \partial_j \phi \; \partial_l A_m. \tag{31}$$

Using the identity (see e.g. Landau and Lifshitz [23, page 18])

$$e_{ijk} e_{lmk} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}, \tag{32}$$

we find

$$u_i = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \partial_j \phi \; \partial_l A_m = \delta_{il} \partial_j \phi \; \partial_l A_j - \delta_{im} \partial_j \phi \; \partial_j A_m, \tag{33}$$

so

$$u_i = \partial_j \phi \; \partial_i A_j - \partial_j \phi \; \partial_j A_i. \tag{34}$$

When integrating this over the volume containing the system we would like to have terms that are divergences, so that we can use Gauss' theorem to turn them into surface integrals. These should then go to zero as the distance to the surface goes to infinity. To achieve this we use that

$$u_i = \partial_j (\phi \; \partial_i A_j) - \phi \; \partial_j \partial_i A_j - \partial_j (\partial_j \phi \; A_i) + (\partial_j \partial_j \phi) A_i. \tag{35}$$

This can be written

$$u_i = \nabla \cdot (\phi \; \partial_i \boldsymbol{A}) - \phi \; \partial_i \nabla \cdot \boldsymbol{A} - \nabla \cdot (A_i \nabla \phi \;) + (\nabla^2 \phi) A_i. \tag{36}$$

Using $\nabla \cdot \boldsymbol{A} = 0$ and $\nabla^2 \phi = -4\pi\varrho$, this becomes

$$u_i = \nabla \cdot (\phi \; \partial_i \boldsymbol{A} - A_i \nabla \phi) - 4\pi \varrho A_i. \tag{37}$$

Integrating this over a volume V contained in a closed surface S, Gauss' theorem gives

$$\int_V u_i \mathrm{d}V = \int_S (\phi \; \partial_i \boldsymbol{A} - A_i \nabla \phi) \cdot \mathrm{d}\boldsymbol{S} - \int_V 4\pi \varrho A_i \mathrm{d}V. \tag{38}$$

For particles and currents in a limited region of space, one should have $\phi \sim 1/r$ and $A_i \sim 1/r^2$ for large distance r from the region. Since these quantities are differentiated in the two terms in the surface integral, these should behave as $\sim 1/r^4$ with distance r. So, making the volume big enough, the surface integrals become negligible, and we have

$$\int \boldsymbol{u} \mathrm{d}V = \int \nabla \phi \times (\nabla \times \boldsymbol{A}) \mathrm{d}V = -\int 4\pi \varrho \boldsymbol{A} \mathrm{d}V \tag{39}$$

when integrating over all space[2].

ORCID iDs

Hanno Essén https://orcid.org/0000-0003-0130-9643

[2] Having derived this result Trammell [44] applies it to the infinite solenoid. One should note, however, that results that are derived assuming that surface integrals are zero at infinity do not necessarily apply to infinite solenoids.

References

[1] Furry W H 1969 Examples of momentum distributions in the electromagnetic field and in matter *Am. J. Phys.* **37** 621–36
[2] Carpenter C J 1989 Comparison of the practical advantages of alternative descriptions of electromagnetic momentum *IEE Proceedings* **136** 101–13
[3] Gsponer A 2007 Electromagnetic momentum of static charge and steady current distributions *Eur. J. Phys.* **28** 1021–42
[4] Griffiths D J 2012 Resource letter EM-1: Electromagnetic momentum *Am. J. Phys.* **80** 7–18
[5] McDonald K T 2006 *Electromagnetic momentum of a capacitor in a uniform magnetic field* E-print: https://www.researchgate.net/publication/251577955_Electromagnetic_Momentum_of_a_Capacitor_in_a_Uniform_Magnetic_Field
[6] Lorrain P 1982 Alternative choice of the energy flow vector of the electromagetic field *Am. J. Phys* **50** 492–492
[7] Yu B and Hu K 2012 Introducing electromagnetic field momentum *Eur. J. Phys.* **33** 873–81
[8] Calkin M G 1966 Linear momentum of quasistatic electromagnetic fields *Am. J. Phys.* **34** 921–5
[9] Calkin M G 1996 *Lagrangian and Hamiltonian Mechanics* (Singapore: World Scientific)
[10] Konopinski E J 1978 What the electromagnetic vector potential describes *Am. J. Phys.* **46** 499–502
[11] Konopinski E J 1981 *Electromagnetic Fields and Relativistic Particles* (New York: McGraw-Hill Book Company, Inc.)
[12] Gingras Y 1980 Comment on 'What the electromagnetic vector potential describes' *Am. J. Phys.* **48** 84–84
[13] Semon M D and Taylor J R 1996 Thoughts on the magnetic vector potential *Am. J. Phys.* **64** 1361–9
[14] Aguirregabiria J M, Hernández A and Rivas M 1990 On dynamical equations and conservation laws in quasistatic electromagnetic systems *Am. J. Phys.* **58** 635–9
[15] Aguirregabiria J M, Hernández A and Rivas M 2004 Linear momentum density in quasistatic electromagnetic systems *Eur. J. Phys.* **25** 555–67
[16] Hnizdo V 1992 Coservation of linear and angular momentum and the interaction of a moving charge with a magnetic dipole *Am. J. Phys.* **60** 242–6
[17] Johnson F S, Cragin B L and Hodges R R 1994 Electromagnetic momentum density and the Poynting vector in static fields *Am. J. Phys.* **62** 33–41
[18] Babson D, Reynolds S P, Bjorkquist R and Griffiths D J 2009 Hidden momentum, field momentum, and electromagnetic impulse *Am. J. Phys.* **77** 826–33
[19] Boyer T H 2015 Interaction of a magnet and a point charge: Unrecognized internal electromagnetic momentum *Am. J. Phys.* **83** 433–42
[20] Redfern F 2017 Magnets in an electric field: hidden forces and momentum conservation *Eur. Phys. J. D* **71** 1–17
[21] Darwin C G 1920 The dynamical motions of charged particles *Philos. Mag. (UK)* **39** 537–51
[22] Jackson J D 1999 *Classical Electrodynamics* 3rd edn (New York: Wiley)
[23] Landau L D and Lifshitz E M 1975 *The Classical Theory of Fields* 4th edn (Oxford: Pergamon)
[24] Page L and Adams N I 1940 *Electrodynamics* (New York: Van Nostrand)
[25] Podolsky B and Kunz K S 1969 *Fundamentals of Electrodynamics* (New York: Marcel Dekker)
[26] Schwinger J, DeRaad L L Jr., Milton K A and Tsai W-y 1998 *Classical Electrodynamics* (Massachusetts: Perseus books, Reading)
[27] Breitenberger E 1968 Magnetic interactions between charged particles *Am. J. Phys.* **36** 505–15
[28] Kennedy F J 1972 Approximately relativistic interactions *Am. J. Phys.* **40** 63–74
[29] Essén H 1996 Darwin magnetic interaction energy and its macroscopic consequences *Phys. Rev.* E **53** 5228–39
[30] Essén H 1999 Magnetism of matter and phase-space energy of charged particle systems *J. Phys. A: Math. Gen.* **32** 2297–314
[31] Stettner R 1971 Conserved quantities and radiation effects for a closed system of charged particles *Ann. Phys. (N.Y.)* **67** 238–51
[32] Boyer T H 2012 Examples and comments related to relativity controversies *Am. J. Phys.* **80** 962–71
[33] Boyer T H 2015 Faraday induction and the current carriers in a circuit *Am. J. Phys.* **83** 263–442
[34] Essén H 2005 Electrodynamic model connecting superconductor response to magnetic field and to rotation *Eur. J. Phys.* **26** 279–85

[35] Essén H 2005 Magnetic dynamics of simple collective modes in a two-sphere plasma model *Phys. of Plasmas* **12** 1–7
[36] Essén H 2009 From least action in electrodynamics to magnetomechanical energy–a review *Eur. J. Phys.* **30** 515–39
[37] Essén H 2011 Classical diamagnetism, magnetic interaction energies, and repulsive forces in magnetized plasmas *EPL* **94** 1–5
[38] Essén H and Fiolhais M C N 2012 Meissner effect, diamagnetism, and classical physics—a review *Am. J. Phys.* **80** 164–9
[39] Darwin C G 1936 Inertia of electrons in metals *Proc. R. Soc. Lond. (UK)* **154A** 61–6
[40] Hnizdo V 2004 Potentials of a uniformly moving point charge in the Coulomb gauge *Eur. J. Phys.* **25** 351–60
[41] Essén H 2007 The exact Darwin Lagrangian *EPL* **79** 1–3
[42] Jackson J D 2002 From Lorenz to Coulomb and other explicit gauge transformations *Am. J. Phys.* **70** 917–28
[43] Berche B, Malterre D and Medina E 2016 Gauge transformations and conserved quantities in classical and quantum mechanics *Am. J. Phys.* **84** 616–25
[44] Trammell G T 1964 Aharonov-Bohm paradox *Phys. Rev.* **134** B1183–1184
[45] Kittel C 1996 *Introduction to Solid State Physics* 7th edn (New York: Wiley)