CHAPTER 4

Wave-Particle Duality

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4.1. Introduction

In this chapter we focus on the wave-like description of a moving particle, especially of an electron. We look at wave-particle duality from three different perspectives, which turn out to be complementary.

Firstly, in section 4.2 we treat the subject from the perspective of general relativity. The mathematics of section 4.2 is more complex than the rest of the book, and assumes that the reader is familiar with general relativity.

We find that there is a one-to-one correspondence between the metrics of general relativity and wave equations, which will permit us to associate the particle momentum with its corresponding quantum mechanical operator. Before entering into the specifics, we first give an overview of some differential geometry, Cifford Algebra and the relationship between them, emphasizing the role of the Principle of Equivalence.

Secondly, in section 4.3 we build upon the results of chapter 3 to develop a geometric interpretation of wave-particle duality: we find that the approach of chapter 3 matches the results of section 4.2.

Finally, in section 4.4 we look at the subject from the perspective of electromagnetic waves' Lorentz transformation, and establish the correspondence between Lorentz transformation rules and the previously derived results. The $\triangle x \triangle p \geq \frac{\hbar}{2}$ quantum mechanical uncertainty rule arises as a consequence of electromagnetic noise background.

4.2. Metrics and the Dirac equation

What is the role of space-time metrics in the description of fundamental forces and interactions? As mentioned in the book introduction, the motion of planets was first described and predicted by epicycloid fomulas. Newton discovered that planetary motion is caused by gravitational forces, given by the $F_{gravity} = G\frac{mM}{r^2}$ formula. While Newton's formula has the same the accuracy as epicycloids for predicting planetary motion, it represents a deeper understanding of elementary interactions and gives a unified method to calculate the motion of planets, falling apples, and rockets. A next level of understanding gravitational forces was brought by Einstein's general relativity theory. As illustrated in figure 4.2.1, general relativity allows us to calculate how mass creates spacetime curvature. In this theory, planets are moving along a straight line in their locally flat spacetime, but this locally straight trajectory becomes a closed ellipsoid loop in the globally curved spacetime. One may calculate approximately the same planetary trajectory via Newton's gravitational force calculation over a flat spacetime and via Einstein's spacetime curvature calculation. Do we gain any new insight through the concepts and equations of general relativity? As also illustrated in figure 4.2.1, general relativity gives a unified method to calculate the motion of planets and the gravitational deflection of electromagnetic waves. Thus we gained new knowledge with respect to Newton's theory, which could not say anything about light deflection.

General relativity also describes how electromagnetic field energy curves spacetime. As a thought experiment, we can let two electrically charged cannon balls orbit around each other in space. We put enough charge onto them, so that gravitational forces are negligible in comparison with electric forces.

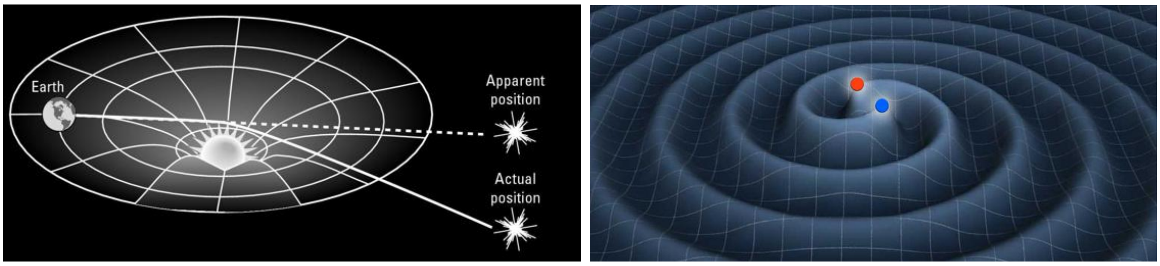

FIGURE 4.2.1. An illustration of gravitational spacetime curvature (left) and electromagnetic spacetime curvature (right).

Applying Maxwell's equations over a flat spacetime, we may calculate the trajectory of these balls spiraling into each other, as well as the radiated electromagnetic wave during this process. Alternatively, we could have could have calculated the space-time curvature, which is not constant in this case, but rather a rotating spiral as shown in figure 4.2.1. Upon lengthy calculations, we would arrive at the same geodesic trajectory result which we got by of applying Maxwell's equations over a flat spacetime. So far there is nothing controversial, and the involved relativistic equations are well known. What happens when the balls are scaled down to microscopic scale? As the ball radius shrinks, electromagnetic fields get more intense, which generates more curved spacetime. Intuitively, the role of spacetime curvature must be immense in quantum mechanics. However, this subject is still unexplored: quantum theorists work exclusively over flat spacetime metric. We remedy this shortcoming, and explore what new insights can be gained by calculating particle geodesics in curved spacetime.

Before even calculating any spacetime curvature, section 2.13 revealed one dramatic consequence of charged particles' free movement along spacetime geodesics: it explains the single-frequency light emission and absorption during quantum mechanical state transitions. In other words, it is at the heart of clarifying over 100 years of misconceptions about the so-called "photon particles".

We begin with an intuitive and non-rigorous approach to our methodology by indicating two ways in which quantum mechanical wave equations can be obtained from the metrics of general relativity, without any explicit recourse to Lagrangians or Hamiltonians. We will then combine the results of the two approaches into a mathematical theorem. In the following section, we will impose more rigorous constraints which will enable us to identify the spinor formulation given here with the usual Hilbert Space formulation of quantum mechanics. Einsteinian notation will be used for index summations throughout this chapter.

4.2.1. Dual Equations. In his book titled *Differential Geometry and Its Applications,* Oprea loosely describes differential geometry as a subject "concerned with understanding shapes and their properties in terms of calculus"[**2**]. These shapes are usually expressed in terms of curves, areas and volumes embedded in locally flat surfaces which are given the technical name of "manifolds". The General Theory of Relativity associates gravitational fields and also electromagnetic fields with four-dimensional differential manifolds. Consequently, the language of differential geometry is at the heart of the subject. We denote a manifold and its local metric tensor as $(\mathcal{M}, g)$. Particles move on curves, which in turn can be defined in terms of metrics $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ where (dx^μ) represents a tangent vector along the curve, expressed in terms of differentials defined with respect to a basis. At each point along the curve, we can also assign a vector field, $(\partial/\partial x^\mu)$, whose operation on a function ψ can be associated with a gradient $\frac{\partial\psi}{\partial x^\mu}$ defined over the manifold. In effect, there is a canonical 1-1 correspondence between vectors defined on the tangent space, associated with the directional derivative along a curve, and those which can be identified as gradients defined over the dual space associated with the same curve:

$$T_0^1(M) \leftrightarrow T_1^0(M) \tag{4.2.1}$$

$$v^\mu \partial_\mu \leftrightarrow \tilde{v}_\mu dx^\mu \tag{4.2.2}$$

Note that this is true for general coordinate systems defined over the manifold. The reader is advised to review sections 0.3-0.4 which introduce the mathematics of exterior algebra and tensors.

Since a manifold is a locally flat surface, we can erect an orthogonal tetrad (vierbien) at each point. In terms of general relativity and the principle of equivalence, this means that there exists such a (local) tetrad at every point of an electromagnetic field that for each $v \in T_0^1$ and the corresponding $\tilde{v} \in T_1^0$ we can write respectively

$$v = v^a \partial_a \qquad \text{and} \qquad \tilde{v} = v_a dx^a$$

The use of tetrads is essential when we work with spinors.

We will now exploit this 1-to-1 correspondence to identify the quantum-mechanical wave equations as the dual of the Dirac "square-root" of the metric (see also [**3**] and [**4**]). Let

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{ab} dx^a dx^b \tag{4.2.3}$$

where a and b refer to local tetrad coordinates and η to a rigid (local) Minkowski metric. Associated with this metric η are the scalar ds and a matrix $\tilde{ds} \equiv \gamma_a dx^a$ such that $\tilde{ds}^2 = ds^2$. Here γ_a are Clifford bases, which transform as a covariant vector under coordinate transformations. Geometrically, $\gamma_a dx^a$ is a vector and ds is its norm. We will work with the 2×2 complex matrix representation of $\gamma_a dx^a$. There are many possible choices for the matrix representation of γ_a, and the most frequently used ones are the Pauli spin matrices introduced in section 2.10.

We define $\{\gamma_a, \gamma_b\} \equiv \gamma_a \gamma_b + \gamma_b \gamma_a$, which leads to the $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$ identity for Clifford bases according to their commutation rules.

Note also that $g_{\mu\nu}(x) = \eta_{ab} e^a_\mu(x) e^b_\nu(x)$ with $e^a_\mu(x)$ forming local tetrads at x, which are the tensor relationships between the two metric tensors. Furthermore, ds can be considered as an "eigenvalue" of the linear operator $\tilde{ds}$ by forming the spinor eigenvector equation $\tilde{ds}\xi = ds\xi$. We introduced such eigenvalue equation already in section 0.7.3, where the spin operator was an equivalent of $\tilde{ds}$. The spinors of section 0.7.3 represented spatial rotation, and thus had the following form:

$$\xi = \xi_x \gamma_y \wedge \gamma_z + \xi_y \gamma_x \wedge \gamma_z + \xi_z \gamma_x \wedge \gamma_y$$

The eigenvalue equation was based on a dual mapping, which we introduced in section 0.3, and where the eigenvalues were encoding the direction of spinning. Such dual mapping works over three dimensional space, where the spinor can rotate around any spatial coordinate.

Based on the above definitions, we associate the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{ab} dx^a dx^b \tag{4.2.4}$$

with the spinor equation:

$$ds\xi = \gamma_a dx^a \xi \tag{4.2.5}$$

This equation, in a natural way, associates spinors directly with the metrics of general relativity. Moreover, in agreement with the general theory of eigenvectors, if ξ is a solution so also is $\psi(z_0, z_1, z_2, z_3)\xi$ where ψ is any complex scalar valued function. Indeed, there is no reason why ψ cannot be an L^2 function that corresponds to a quantum-mechanical wave function. It is also worth keeping in mind that every operator has its corresponding eigenfuctions and by definition, these eigenfuctions are invariant with respect to the action of the operator. While in quantum mechanics the act of measurement interferes with the initial state of the wave function, the eigenvector corresponds to the invariant state with respect to the action of the operator. Usually this is framed in the language of the projection postulate and in this context quantum theorists talk about a collapsed wave function. However, this is a misnomer. In reality, it is not that the wave function has collapsed, but rather it has been transformed into another one, with the eigenvector still remaining invariant. For example, if we consider the rotation of the earth, the eigenvector will correspond to its axis of rotation while all other points are constantly in motion because of the rotation.

Recalling the definitions of section 0.3, each vector $\frac{\partial}{\partial x^a}$ can be mapped to a dual one-form dx^a. In a similar way, the $\tilde{ds}$ matrix above can be seen as the dual of the expression $\tilde{\partial}_s \equiv \gamma^a \frac{\partial}{\partial x^a}$, where γ^a is defined by the relationship $\{\gamma^a, \gamma_b\} = 2\delta^a_b$, or equivalently $\eta_{ab}\gamma^a = \gamma_b$.

If we let s describe the length of a particle's trajectory along a curve $(x^0(s), x^1(s), x^2(s), x^3(s)) \in (\mathcal{M}, g)$ then s can be regarded as an independent parameter with an associated 1-form ds, which is the dual of the tangent vector ∂_s. Note that in terms of the basis vectors for $T_p(\mathcal{M})$ and $T_p^*(\mathcal{M})$ we can write $\partial_s = \frac{\partial x^a}{\partial s}\partial_a$ and $ds = \frac{\partial s}{\partial x^a} dx^a$. It also follows that this dual map is given by $ds\partial_s \equiv \frac{\partial s}{\partial x^i}\frac{\partial x^i}{\partial s} = 1$. Putting these two results together allows us to consider equation 4.2.5 as the dual of the equation:

$$\frac{\partial \psi}{\partial s} = \gamma^a \frac{\partial \psi}{\partial x^a}, \tag{4.2.6}$$

where $\frac{\partial}{\partial s}$ refers to differentiation along a curve parametrized by s. We shall refer to equation 4.2.6 as a (generalized) Dirac equation and later show how it relates to the usual form of this equation. At times, we shall also refer loosely to it as a "dual wave-equation". **This 1-to-1 map between equations 4.2.5 and 4.2.6 formally defines what we mean by wave-particle duality.** They both go together and one cannot exist without the other. The metric corresponds to the particle property and the wave equation to its wave property. Given the metric, we can write down the wave and conversely, given the wave we can write down the metric. It is only a matter of convention whether we begin first with the wave and then the metric or vice-versa. It remains to better understand the origin and properties of this wave from a physical point of view.

We can now compare our approach against the traditional Dirac theory. The traditional Dirac theory approach is illustrated in table 1: it is based on making an assignment between the wavefunction derivatives and the eigenvalues shown in the last column. As discussed in section 2.10, the traditional Dirac equation is an eigenvalue equation corresponding to the linearized $\epsilon^2 - (pc)^2 = (mc^2)^2$ energy-momentum relationship. Essentially, the traditional Dirac theory is a "guessing" of the eigenvalue assignments. Regarding the mc^2 eigenvalue, the corresponding differentiaton operator was not guessed: instead of the corresponding operator, the $m\psi$ eigenvalue terms is placed directly into the Dirac equation. This hybridization has two consequences: i) the other terms get multiplied by $i\hbar$ in order to compensate the operator-to-eigenvalue substitution, and ii) the resulting Dirac equation is no longer a local equation due to the presence of the m coefficient, which represents electromagnetic energy integration over the whole space. While mainstream theorists consider the traditional Dirac equation to be their fundamental starting point, we observed already in chapter 2 that it is not a proper field equation because m is neither a physical constant or a local parameter. We gave the example of electron-positron annihilation process, during which the non-local particle masses disappear completely.

In contrast, we work on a more fundamental level with the metrics of space-time. Our generalized Dirac equation is a local equation, and it is therefore the correct foundation for building a relativistically consistent field theory. The spinor appearing in our equation is nothing else than the Zitterbewegung of electromagnetic field, which we discussed in chapter 3. Under this approach, the wave-particle duality is no longer a mystery, but a mathematical consequence of duality transformations. By avoiding the above-mentioned hybridized Dirac equation, we avoid all the wavefunction interpretation difficulties which plagued quantum mechanics over the past 100 years. Instead of guessing the eigenvalues shown in the last column of table 1, we will use Hamilton-Jacobi theory to derive them.

	Space-time metrics	Dual of the metric	Dirac's assignment
Length	ds	$\frac{\partial\psi}{\partial s}$	mc^2
Time difference	dt	$\frac{\partial\psi}{\partial t}$	ϵ
Spatial distance	dx	$\frac{\partial\psi}{\partial x}$	pc
Eigenvalue eq.	$ds\xi = \gamma_a dx^a \xi$	$\frac{\partial\psi}{\partial s} = \gamma^a \frac{\partial\psi}{\partial x^a}$	$m\psi = i\hbar\gamma^a \frac{\partial\psi}{\partial x^a}$

TABLE 1. The relationship between space-time metrics and Dirac's original assignments

There are also algebraic properties related to the duality structure that we explore in the following section. These will help us better understand the physics.

4.2.2. Clifford Algebra Properties. Recall the Clifford multiplication rule given by equation 0.2.5. Using the symmetry property of $\boldsymbol{u}\cdot\boldsymbol{v}$ and the antisymmetry property of $\boldsymbol{u}\wedge\boldsymbol{v}$, the Clifford product of $\boldsymbol{u}$ and $\boldsymbol{v}$ vectors can be written as

$$2\boldsymbol{u}\boldsymbol{v} = \{\boldsymbol{u},\boldsymbol{v}\} + [\boldsymbol{u},\boldsymbol{v}]\,. \tag{4.2.7}$$

Consider the two operators $\tilde{ds}$ and $\tilde{\partial}_s$, which are matrix representations of Clifford vectors. On taking their Clifford product we find that

$$\begin{aligned} 2\tilde{ds}\,\tilde{\partial}_s\psi &= \{\tilde{ds},\tilde{\partial}_s\psi\} + [\tilde{ds},\tilde{\partial}_s\psi] && (4.2.8)\\ &= 2\frac{\partial\psi}{\partial s}ds + [\tilde{ds}\psi,\tilde{\partial}_s\psi] && (4.2.9)\end{aligned}$$