quantum mechanical energy operator $i\hbar\partial_t$ instead of $i\frac{\hbar}{2}\partial_t$? Furthermore, an absorbed quantum unit of electromagnetic energy also carries quantized angular momentum; a photon's spin angular momentum is measured to be quantized at $\pm\hbar$. The angular momentum and rotational phase are conjugate variables, as discussed in reference [**3**]. The angular momentum of a circularly polarized wave quantum may be $\pm\frac{\hbar}{2}$. Why does a photon carry then $\pm\hbar$ angular momentum? We return to these questions in section 2.13.

2.7. The electromagnetic frequency of a massive particle

When an electron and a positron annihilate, the produced electromagnetic radiation is perceived as two photons, each carrying $\hbar\omega = mc^2$ energy and $\hbar$ angular momentum. By conservation of energy and angular momentum, an electron must comprise then an electromagnetic field which contains the same amount of energy and angular momentum, $\hbar\omega$ and $\hbar$ respectively. What is the electromagnetic topology difference between a transversal electromagnetic wave and an elementary particle?

In the above discussed transversal electromagnetic wave, the electric and magnetic fields have locally the same magnitude, induce each other via rotating field orientation, and the wave has zero rest mass. The electron however carries an elementary charge, therefore its scalar field is non-zero. The circularly polarized electromagnetic field is now described by $G = S + F = (S, \mathbf{0}) + (0, \mathbf{B}\mathbf{e}_{xyz} - i\mathbf{E}\mathbf{e}_t)$. As derived in chapter 1, both the scalar and transversal field components move at the speed of light. In analogous way to the existence of F_+ and F_- electromagnetic fields, there must be two types of scalar electromagnetic fields. The corresponding G_+ and G_- electromagnetic fields take the following form for fields with electric charges:

- $F_{l+} = (0, \mathbf{B}\mathbf{e}_{xyz} - i\mathbf{E}\mathbf{e}_t) \to G_{l+} = (S, \mathbf{B}\mathbf{e}_{xyz} - i\mathbf{E}\mathbf{e}_t)$
- $F_{l-} = (0, i\mathbf{B}\mathbf{e}_{xyz} + \mathbf{E}\mathbf{e}_t) \to G_{l-} = (iS, i\mathbf{B}\mathbf{e}_{xyz} + \mathbf{E}\mathbf{e}_t)$
- $F_{r+} = (0, \mathbf{B}\mathbf{e}_{xyz} + i\mathbf{E}\mathbf{e}_t) \to G_{r+} = (-S, \mathbf{B}\mathbf{e}_{xyz} + i\mathbf{E}\mathbf{e}_t)$
- $F_{r-} = (0, i\mathbf{B}\mathbf{e}_{xyz} - \mathbf{E}\mathbf{e}_t) \to G_{r-} = (-iS, i\mathbf{B}\mathbf{e}_{xyz} + \mathbf{E}\mathbf{e}_t)$

In the above expressions, the sign of the field S determines whether the charge is positive or negative; i.e. it has opposite sign for a particle and its anti-particle.

Using the results of chapter 1, Maxwell's equations retain their usual structure:

$$D_l G_{l+} = 0,\ D_r G_{r+} = 0 \tag{2.7.1}$$

$$D_l G_{l-} = 0,\ D_r G_{r-} = 0 \tag{2.7.2}$$

The energy and momentum densities of a massive particle are now given by the $N_+ = \frac{1}{2}G_+\mathbf{e}_t\widetilde{G_+}$ expression, which replaces the $\frac{1}{2}F_+\mathbf{e}_t\widetilde{F_+}$ expression of the scalar-free field. We evaluate the electromagnetic energy-momentum:

$$\begin{aligned}
N_{l+} &= \frac{1}{2}\left(G_{l+}\mathbf{e}_t\right)\widetilde{G_{l+}} = -\frac{1}{2}(S\mathbf{e}_t, i\mathbf{B} + i\mathbf{E})(-S, \mathbf{B}\mathbf{e}_{xyz} + \mathbf{E}\mathbf{e}_{xyz}) = \\
&= \frac{1}{2}S^2\mathbf{e}_t - \frac{1}{2}S\left(\mathbf{e}_t\left(\mathbf{B} + \mathbf{E}\right)\mathbf{e}_{xyz} - i\left(\mathbf{B} + \mathbf{E}\right)\right) - \frac{1}{2}i\left(\mathbf{B} + \mathbf{E}\right)\left(\mathbf{B} + \mathbf{E}\right)\mathbf{e}_{xyz} = \\
&= \frac{1}{2}\left(S^2 + E^2 + B^2\right)\mathbf{e}_t - \frac{1}{2}S\left(-\left(\mathbf{B} + \mathbf{E}\right)i - i\left(\mathbf{B} + \mathbf{E}\right)\right) + \frac{1}{2}i2\mathbf{B}\times\mathbf{E} = \\
&= \frac{1}{2}\left(S^2 + E^2 + B^2\right)\mathbf{e}_t + i\mathbf{B}\times\mathbf{E}
\end{aligned} \tag{2.7.3}$$

$$\begin{aligned}
N_{r+} &= \frac{1}{2}\left(G_{r+}\mathbf{e}_t\right)\widetilde{G_{r+}} = -\frac{1}{2}(S\mathbf{e}_t, i\mathbf{B} - i\mathbf{E})(-S, \mathbf{B}\mathbf{e}_{xyz} - \mathbf{E}\mathbf{e}_{xyz}) = \\
&= \frac{1}{2}\left(S^2 + E^2 + B^2\right)\mathbf{e}_t - i\mathbf{B}\times\mathbf{E}
\end{aligned} \tag{2.7.4}$$

The above equations correspond to the energy and momentum densities of the G_+ electromagnetic field. Analogously, the energy and momentum of the G_- electromagnetic field is given by the $N_- = -\frac{1}{2}G_-\mathbf{e}_t\widetilde{G_-}$ expression.

Equations 2.7.3 and 2.7.4 imply that the Poynting-vector part of the electromagnetic field is the well-known $\mathbf{B} \times \mathbf{E}$ formula, regardless of the scalar field's presence or absence. This result deviates from the $\mathbf{B} \times \mathbf{E} + S\mathbf{E}$ formula obtained in chapter 1. This difference stems from the different time variable choice in the $A = (iA_t\mathbf{e}_t, \mathbf{A})$ definition, which we use in this chapter. The $A = (iA_t\mathbf{e}_t, \mathbf{A})$ choice is remarkable because i) the Poynting-vector formula remains same as in the scalar-free case, and ii) the energy and momentum density formulas are symmetric in terms of E and B fields.

According to the Heisenberg uncertainty principle, when the G_+ or G_- field oscillates at a certain frequency ω, it has at least $\frac{\hbar\omega}{2}$ energy and $\frac{\hbar}{2}$ angular momentum. However, the electron-positron annihilation process demonstrates that the electron carries $\hbar\omega$ energy and $\hbar$ angular momentum. In this way, the electron-positron pair satisfies energy and angular momentum conservation with respect to the transversal wave from which it is created or into which it is annihilated. These values are a starting point in the search for an analytic solution of a particle's internal fields.

Since S, E, and B are all derived from the same electromagnetic vector potential, they all oscillate at the same frequency. This internal frequency is related to the particle mass via the $\hbar\omega = mc^2$ formula. In chapter 3 we shall explicitly derive the relativistic increase of particle mass under Lorentz boost, i.e. we will prove the equivalence between the relativistic increase of particle mass and the relativistic shift in the particle's electromagnetic oscillation frequency. Here, we just note a pertinent point about mass conservation. In a thought experiment, we place an electron and a positron into an empty box, and initially both particles are at rest. As the two particles speed towards each other due to electric attraction, they gain kinetic energy. Where is this energy coming from? There is no other source for it than the electric field of the particles. As the two particles get closer, there is more and more cancellation between the electric fields of the two particles, and the overall electric field energy is reduced. In other words, the energy of the charges' extended electric field is gradually converted into the kinetic energy of the particles. As the particles gain kinetic energy, their relativistic mass increases. From the perspective of the whole box, there was no outside influence, so its total mass must remain invariant. Therefore the mass gained by the kinetic boost must be exactly counterbalanced by the mass lost by electric field cancellations. This thought experiment is a direct illustration of the electromagnetic field origin of particle mass. One must refrain from thinking about a particle being at a certain point in space, since the particle is represented by the whole electromagnetic wave.

The main result from this section is the $\hbar\omega = mc^2$ relationship between the massive particle and its internal electromagnetic frequency. If the particle's electromagnetic field solution involves a circular rotation of its charge, the angular frequency of this rotation must be $\omega = \frac{mc^2}{\hbar}$. This angular frequency is the so-called "De Broglie frequency". Although the $\omega = \frac{mc^2}{\hbar}$ frequency is very high, the authors of [**8**] succeeded in experimentally measuring it a few years ago.

2.8. The symmetries of electromagnetism

In this section we look into the meaning of using real versus i-multiplied time coordinate. In chapter 1, we defined the electromagnetic field and the space-time differential using real coordinates:

- $A_{l+} = (A_t\mathbf{e}_t, \mathbf{A})$, $D_l = (-\partial_t\mathbf{e}_t, \nabla)$
- $A_{r+} = (-A_t\mathbf{e}_t, \mathbf{A})$, $D_r = (\partial_t\mathbf{e}_t, \nabla)$
- $A_{l-} = T_l = (iA_t\mathbf{e}_t, i\mathbf{A})$, $D_l = (-\partial_t\mathbf{e}_t, \nabla)$
- $A_{r-} = T_r = (-iA_t\mathbf{e}_t, i\mathbf{A})$, $D_r = (\partial_t\mathbf{e}_t, \nabla)$

In the above expression, the T symbol refers to the vector potential which generates magnetic charges and currents, as introduced in section 1.3.7. In this chapter, we introduced the $\mathbf{e}_t \to i\mathbf{e}_t$ transformation of the time-wise coordinate:

- $A'_{l+} = (iA_t\mathbf{e}_t, \mathbf{A})$, $D'_l = (-i\partial_t\mathbf{e}_t, \nabla)$
- $A'_{r+} = (-iA_t\mathbf{e}_t, \mathbf{A})$, $D'_r = (i\partial_t\mathbf{e}_t, \nabla)$
- $A'_{l-} = (-A_t\mathbf{e}_t, i\mathbf{A})$, $D'_l = (-i\partial_t\mathbf{e}_t, \nabla)$
- $A'_{r-} = (A_t\mathbf{e}_t, i\mathbf{A})$, $D'_r = (i\partial_t\mathbf{e}_t, \nabla)$

We verified explicitly that Maxwell's equations remain the same regardless of how we choose the time-wise basis. Electromagnetism has a symmetry with respect to this coordinate transformation. In fact it is not a discrete, but a continuous symmetry because we can generalize the choice of time coordinate transformation to any angle according to the $\mathbf{e}_t \to e^{i\theta}\mathbf{e}_t$ choice. Even though i is the Clifford pseudo-scalar, this generalization is still valid because the exponentiation of the Clifford pseudo-scalar is a well defined operation in Clifford algebra.

Let's review briefly the various electromagnetic symmetries. Firstly, Lorentz transformations of a reference frame are accomplished via spatial rotations and boosts. The Lorentz group of reference frame transformations is given by the $SU(2) \times SU(2)$ group, corresponding to a rotation of axial orientation and a boost into a certain direction. Here, we used the fact that the usual $SO(3)$ rotational symmetry of space is double covered by the $SU(2)$ group; this result was derived in the mathematical preliminaries chapter. It is interesting to note that the space rotation type $SU(2)$ symmetry preserves $E^2 + B^2$, while Lorentz boost type type $SU(2)$ symmetry preserves $E^2 - B^2$.

By comparing equation 1.3.29 with equations 2.7.3 and 2.7.4, we observe the same expression for the electromagnetic energy density: $\frac{1}{2}\left(S^2 + E^2 + B^2\right)\mathbf{e}_t$. Although the $\mathbf{e}_t \to e^{i\theta}\mathbf{e}_t$ transformation changes the momentum density part, it preserves the electromagnetic energy density. Considering θ as yet another rotation angle, electromagnetic energy density has a $U(1)$ type rotation-like symmetry with respect to this angle.

Physical rotations of space also preserve electromagnetic energy density. Therefore, the above-defined $\mathbf{e}_t \to e^{i\theta}\mathbf{e}_t$ rotation plus spatial rotations form the $U(1) \times SU(2)$ group of transformations, which preserves the electromagnetic energy density. The study of electromagnetic symmetries is a useful tool for identifying conserved quantities.

Rotational symmetries of space correspond to angular momentum conservation. The $\mathbf{e}_t \to e^{i\theta}\mathbf{e}_t$ rotational symmetry also corresponds to the conservation of a certain electromagnetic oscillation mode. Such oscillation mode may or may not be present in a given elementary particle state. Since the nuclear capture and emission of electrons is always accompanied by the capture or emission of neutrinos, it is possible that neutrinos carry a conserved oscillation, corresponding to the $\mathbf{e}_t \to e^{i\theta}\mathbf{e}_t$ symmetry. This hypothesis will be investigated step-by-step throughout the book.

2.9. The longitudinal electromagnetic wave

In section 2.5 we described the simplest spatial solutions of Maxwell's equations, which correspond to scalar-free transversal waves. We wrote the equations for circularly polarized modes, which carry the conserved angular momentum of space rotation symmetry. Considering the $\mathbf{e}_t \to e^{i\theta}\mathbf{e}_t$ symmetry, can we find a wave solution which is the correspoding electromagnetic oscillation mode? A continuous $\mathbf{e}_t \to e^{i\theta}\mathbf{e}_t$ type "rotation" means an oscillation between the electric and magnetic type scalar field. The corresponding spatial solution of Maxwell's equations is a longitudinal wave:

$$E_z = E_0 \sin(\omega t - kz)\,,\; B_z = B_0 \cos(\omega t - kz) \tag{2.9.1}$$

$$S = S_0 \cos(\omega t - Kz) - iS_0 \sin(\omega t - kz) \tag{2.9.2}$$

where $S_0 = E_0 = B_0$ in natural units, and the wave is propagating into the z direction. If this wave had a certain size in the $x-y$ plane, there would be $x-y$ components of the electric and magnetic fields along the edges. However, in the simple plane wave case the electric and magnetic fields have no $x-y$ components. This basic longitudinal wave solution is illustrated in figure 2.9.1.

Considering the $S_0 = E_0 = B_0$ condition of the longitudinal wave solution, the $\frac{1}{2}\left(S^2 + E^2 + B^2\right)\mathbf{e}_t$ electromagnetic energy density formula implies that $\frac{1}{2}$ of the electromagnetic field energy is carried by the scalar field, while $\frac{1}{4}$ of the field energy is carried by the electric field and $\frac{1}{4}$ of the field energy is carried by the magnetic field. This is in contrast to the transversal transversal wave solution, which is scalar-free.

We conclude this section by a summary of electromagnetic wave polarization modes. We identified three principal electromagnetic wave modes in this chapter, as listed in table 1. Each polarization mode has a trivial spatial solution, which may be further classified e.g. according to left/right handed chirality. Since only one of these principal modes has a scalar-free trivial solution, historically only the A_+ mode was the focus of electromagnetism studies.

Despite its apparent simplicity, Maxwell's equation admits rather complex spatial solutions even under the $S = 0$ restriction, such as Laguerre–Gaussian beams or cylindrical Bessel beams. The mathematical form of spatial solutions also depends on the boundary conditions: electromagnetic