A Lie-Admissible Model for Dissipative Plasmas (*).

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1. – In the present note we introduce a model for dissipative plasmas constructed in terms of an elementary generalization of well-known basic equations for plasma physics to dissipative conditions.

For this purpose, let us first recall the role of Hamiltonian mechanics for deriving the Liouville equation of a conservative system.

We consider an n-component system in phase space represented by the Hamiltonian $H(q_k, p_k)$, the density $\varrho(q_k, p_k, t)$ and the velocity $\boldsymbol{V} \equiv (\mathrm{d}q_k/\mathrm{d}t, \mathrm{d}p_k/\mathrm{d}t)$, where $k = 1, 2, \ldots, n$. On account of the property

$$\boldsymbol{\nabla}\cdot\boldsymbol{V} \equiv \frac{\partial}{\partial q_k}\frac{\mathrm{d}q_k}{\mathrm{d}t} + \frac{\partial}{\partial p_k}\frac{\mathrm{d}p_k}{\mathrm{d}t} = \frac{\partial^2 H}{\partial q_k \partial p_k} - \frac{\partial^2 H}{\partial p_k \partial q_k} = 0\,, \tag{1}$$

where summation over repeated indices is assumed, and by defining the Poisson bracket for any pair of functions $A(q_k, p_k)$ and $B(q_k, p_k)$ according to

$$[A, B]_p = \frac{\partial A}{\partial q_k}\frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial B}{\partial q_k}\,, \tag{2}$$

the Liouville equation for our system is given by

$$\frac{\partial \varrho}{\partial t} + \boldsymbol{V}\cdot\boldsymbol{\nabla}\varrho = \frac{\partial \varrho}{\partial t} + [\varrho\cdot H]_p = 0\,. \tag{3}$$

From eqs. (1) and (2) we clearly see the basic role of Hamiltonian mechanics for deriving eq. (3). This implies as a consequence that the Liouville equation (3) possesses well-defined limits of validity, since Hamiltonian mechanics is applicable in its general formulation to conservative (or holonomic) systems and is extendible only to a few

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(1) In connection with the limits of validity of Hamiltonian mechanics see, for instance, H. GOLDSTEIN: *Classical Mechanics* (Reading, Mass., 1965).

specific dissipative (or nonholonomic) systems (1). For instance, when there is a dissipative condition which cannot be expressed in terms of a generalized potential, Hamiltonian mechanics is not longer valid, and the problem of the same basic procedure for deriving an explicit form of the Liouville equation has not been fully clarified yet (2).

2. – In a recent paper (3), an algebraic criterion for selecting a generalization of Hamiltonian mechanics for dissipative systems has been introduced. This criterion can be essentially summarized as follows. The Poisson bracket (2) can be considered, from an algebraic viewpoint, as an abstract product

$$A \cdot B = [A, B]_p , \tag{4}$$

which defines a Lie algebra L. If one considers a dissipative framework, then a generalization of Hamiltonian mechanics generally implies an enlargment of the basic product according to an extended bracket

$$A \circ B = (A, B) \tag{5}$$

which in principle defines no longer a Lie algebra.

The above criterion for selecting enlarged procedures restricts the algebras defined by (5) to the so-called Lie-admissible algebras. These are nonassociative algebras U with abstract product $A \circ B$ such that the attached algebras U^-, which are the same vector spaces as U, but with the new product

$$[A, B]_U = A \quad B - B \cdot A , \tag{6}$$

are Lie algebras.

More explicitly, the above algebraic criterion restricts the possibile generalized formulations by means of the condition on the enlarged bracket

$$(A, B) - (B, A) = c[A, B]_p , \tag{7}$$

where c is a scalar ($\neq 0$ and $\neq \infty$) with respect to q_k and p_k.

Some features of this selectivity criterion are:

1) The Hamiltonian procedure itself satisfies the requirements as basic formulation since

$$[A, B]_p - [B, A]_p = 2[A, B]_p . \tag{8}$$

2) The generalized procedure reduces to the Hamiltonian mechanics when the system becomes conservative, since a Lie-admissible algebra can contract itself to a Lie algebra (4).

(2) For a large bibliography on the generalizations of the Hamilton and Lagrange formulations since 1873 see, W. D. MACMILLAN: *Dynamics of rigid bodies* (New York, 1936).

(3) R. M. SANTILLI: *Dissipativity and Lie-admissible algebras*, Coral Gables preprint no. CTS/M/67/2. To appear in *Meccanica* (April 1969).

(4) R. M. SANTILLI: *Suppl. Nuovo Cimento*, **6**, 1225 (1968).

3) A well-defined content of a system in a conservative condition can be preserved for its extension to dissipative conditions since the original Lie algebra invariance L is « imbedded » ([5]) in the enlarged algebra U.

Among the generalizations of the Hamiltonian mechanics for dissipative systems satisfying the above restriction we consider the so-called pseudo-Hamiltonian mechanics ([6]) which, in terms of two-parameter formulation can be introduced by means of the equations ([3])

$$\begin{cases} \dot{q}_k = \lambda \dfrac{\partial H}{\partial p_k}, \\ \dot{p}_k = \mu \dfrac{\partial H}{\partial q_k} + f_k, \end{cases} \qquad k = 1, 2, \ldots, \mu, \tag{9}$$

where λ and μ are free parameters ($\neq 0$ and $\neq \infty$) independent of q_k and p_k, and f_k are the components of the external forces.

A physical interpretation of the parameters λ and μ can be introduced as follows. Consider a (discrete) conservative system described by the Lagrangian and Hamiltonian

$$\begin{cases} L = T(\dot{q}) - V(q), \\ H = T(p) + V(q), \end{cases} \qquad p = \frac{\partial L}{\partial \dot{q}}, \tag{10}$$

where, as usual, T and V represent respectively the kinematic and potential energies. Suppose that the system for a given period of time becomes dissipative under the action of an external force. Then the kinetic and potential energies generally vary and we can write as true Lagrangian and Hamiltonian for the dissipative conditions

$$\begin{cases} L_D = T_D(q) - V_D(q) + \text{external terms}, \\ H_D = T_D(p) + V_D(q) + \text{external terms}. \end{cases} \tag{11}$$

The above quantities can always be written in terms of the original kinetic and potential energies as

$$\begin{cases} L_D = \lambda T(\dot{q}) + \mu V(q) + \ldots, \\ H_D = \lambda T(p) - \mu V(q) + \ldots, \end{cases} \tag{12}$$

where the quantities

$$\lambda = \frac{T_D}{T}, \qquad \mu = -\frac{V_D}{V} \tag{13}$$

represent the variations of the kinetic and potential energies with respect to the corresponding values for conservative conditions and can be considered, generally, as functions of q, $\dot{q}$, and t, *e.g.* $\lambda = \lambda(\dot{q}, t)$ and $\mu = \mu(q, t)$.

Suppose now, as a « perturbative approach », that either the variations (13) of T and of V are small enough to be approximated by constants, or that the true varia-

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([5]) R. M. SANTILLI: *Nuovo Cimento*, **51** A, 570 (1967).
([6]) R. J. DUFFIN: *Arch. Rational Mech. Anal.*, **9**, 309 (1962).

tions (13) can be approximated by some suitable chosen averages in phase space. Then, because of the independence (7) of λ and μ from q, $\dot{q}$, and t, the pseudo-Lagrange equations are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_D}{\partial \dot{q}_k} - \frac{\partial L_D}{\partial q_k} = \lambda \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} + \mu \frac{\partial L}{\partial q_k} = f_k , \qquad k = 1, 2, \ldots, \mu , \tag{14}$$

and the pseudo-Hamiltonian equations (9) follow as

$$\begin{cases} \dot{q}_k = \dfrac{\partial H_D}{\partial p_k} = \lambda \dfrac{\partial H}{\partial p_k} & \left(p_k = \dfrac{\partial L_D}{\partial q_k} = \lambda \dfrac{\partial L}{\partial q_k} \right) , \\ \dot{p}_k = -\dfrac{\partial H_D}{\partial p_k} + f_k = \mu \dfrac{\partial H}{\partial q_k} + f_k , & k = 1, 2, \ldots, \mu . \end{cases} \tag{15}$$

In this formulation the total time derivative of any function $F(q_k, p_k, t)$ is given by

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \left(\lambda \frac{\partial F}{\partial q_k}\frac{\partial H}{\partial p_k} + \mu \frac{\partial H}{\partial q_k} \right) \qquad (\text{for } f_k = 0) . \tag{16}$$

By recalling that we are in a dissipative condition, the total energy is not conserved and its variation in time is represented by

$$\frac{\mathrm{d}H_D}{\mathrm{d}t} = (\lambda + \mu)\frac{\partial H}{\partial q_k}\frac{\partial H}{\partial p_k} \neq 0 , \qquad \text{for } \lambda \neq -\mu . \tag{17}$$

Finally, from (16), the generalized bracket is given by

$$(A, B) = \lambda \frac{\partial A}{\partial q_k}\frac{\partial B}{\partial p_k} + \mu \frac{\partial A}{\partial p_k}\frac{\partial B}{\partial p_k} , \tag{18}$$

where A and B are any functions of q_k and of p_k.

We clearly see that pseudo-Hamiltonian mechanics satisfies the above selectivity criterion since the algebra $[U(\lambda, \mu)]^-$ characterized by the product (18) is a Lie algebra on account of the property (8)

$$(A, B) - (B, A) = (\lambda - \mu)[A, B]_p . \tag{19}$$

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(7) In this formulation the «isotropy» of the dissipativity in phase space is expressed by the independence of the λ and μ parameters on the degrees of freedom. A further extension of the formulation for anisotropic conditions can be investigated by associating different parameters λ_k and μ_k (with $k = 1, 2, \ldots, n$) with each pair of canonical variables q_k and p_k.

(8) Let us also note that from an algebraic viewpoint only one parameter is essential, since the algebra $U(\lambda, \mu)$ characterized by the product (18) is isomorphic to the so-called isotropic algebra with product

$$(A * B) = \frac{\partial A}{\partial q_k} * \frac{\partial B}{\partial p_k} + \frac{1}{\varepsilon}\frac{\partial B}{\partial q_k} * \frac{\partial A}{\partial p_k} ,$$

where

$$\frac{\partial A}{\partial q_k} * \frac{\partial B}{\partial p_k} = \lambda \frac{\partial A}{\partial q_k}\frac{\partial B}{\partial p_k} , \qquad \frac{\partial B}{\partial q_k} * \frac{\partial A}{\partial p_k} = \lambda \frac{\partial B}{\partial q_k}\frac{\partial A}{\partial p_k} ,$$

and $\varepsilon = \lambda/\mu$. Thus, from an algebraic viewpoint, the two-parameter formulation (9) is equivalent to the Duffin formulation (9).

Furthermore, the procedure itself reduces to Hamiltonian mechanics when the parameters λ and μ obtain the values $+1$ and -1, respectively.

An interesting aspect of this generalized procedure is that the true Hamiltonian H for a conservative region is formally preserved in a dissipative region, where it assumes the name of pseudo-Hamiltonian, since the dissipativity of the system is represented by the parameters involved in the formulation.

Pseudo-Hamiltonian mechanics has been introduced for representing some dissipative electrical network and for describing small motions of dissipative systems about a position of static equilibrium ([6]). It has been also used for introducing a model of dissipative field theory by performing an extension to continuous system ([3]), for whose purpose the above selectivity criterion was derived.

In the following we shall consider the above « perturbative approach » where λ and μ are constants.

3. – In connection with plasma physics, pseudo-Hamiltonian mechanics could be of some interest either for investigating plasma in dissipative conditions because of exchange of energy with an external system (*e.g.* an electric or a magnetic field), or for investigating partial regions of globally conservative plasmas with internal exchange of energy (or of momentum or of angular momentum), whenever the dissipativity does not allow the use of Hamiltonian mechanics (*e.g.* when the dissipative forces are not expressible in terms of generalized potentials ([1]).)

Let us consider a dissipative plasma according to one of the above conditions. The generalization of the Liouville equation in terms of pseudo-Hamiltonian mechanics is given by

$$\frac{\partial \varrho}{\partial t} + (\varrho, H) + (\lambda + \mu)\,\varrho\,\frac{\partial^2 H}{\partial q_k\,\partial p_k} = C\,, \tag{20}$$

where (ϱ, H) is now the bracket (18), the third term in the l.h.s. comes from the contribution of $\nabla \cdot V$ which is now no longer zero as in (1), and C is a collision term.

Let the pseudo-Hamiltonian H be of linear velocity force type, *i.e.*

$$\frac{\partial^2 H}{\mathrm{d}q_k\,\partial p_k} = K\,, \tag{21}$$

with K a constant. Then, by (20) and by computing the total time derivative of ϱ we get (for $C = 0$)

$$\varrho = \varrho_0 \exp\left[-At\right], \tag{22}$$

where $A = K(\lambda + \mu)$. Thus, under the above conditions, the particle density in phase space is no longer constant, but it decreases or increases in time corresponding to $K(\lambda + \mu) > 0$ or $K(\lambda + \mu) < 0$, respectively. For $\lambda = -\mu = 1$, *i.e.* when the pseudo-Hamiltonian mechanics reduces to the Hamiltonian mechanics, we recover the constant behavior in time of ϱ for conservative conditions.

Let us consider a collision term of the form

$$C = -\frac{\varrho - \varrho_0}{t_c}\,, \tag{23}$$

where ϱ_0 is the equilibrium distribution and t_c is the relaxation time. Then the generalized Fokker-Planck equation in terms of pseudo-Hamiltonian mechanics is given by

$$\frac{\partial \varrho}{\partial t} + \boldsymbol{V}\cdot\boldsymbol{\nabla}_q \varrho + \boldsymbol{F}\cdot\boldsymbol{\nabla}_p \varrho = -\frac{X\varrho - \varrho_0}{t_c}\,, \tag{24}$$

where

$$\boldsymbol{V} = \dot{\boldsymbol{q}}\,, \qquad \boldsymbol{F} = \dot{\boldsymbol{p}}\,, \qquad X = 1 + t_c K(\lambda + \mu)\,. \tag{25}$$

If the above equation refers to an electron gas under the action of an electric field $\boldsymbol{E}$, then for small deviations from the equilibrium configuration we can write

$$\varrho = \frac{1}{X}\,\varrho_0 - \frac{t_c}{X^2}\,(\boldsymbol{V}\cdot\boldsymbol{\nabla}_q \varrho_0 + e\boldsymbol{E}\cdot\boldsymbol{\nabla}_p \varrho_0)\,, \tag{26}$$

so that to first order, by assuming ϱ_0 to be independent of q_k, we have

$$\varrho = \frac{1}{X}\,\varrho_0 - \frac{t_c}{X^2}\,e\boldsymbol{E}\cdot\boldsymbol{\nabla}_p \varrho_0\,. \tag{27}$$

Under the above assumptions, the electric current and the conductivity tensor defined for conservative conditions by

$$J_K = -\frac{e^2 t_c}{m}\,E_i \int V_K \frac{\partial \varrho_0}{\partial V_i} \tag{28}$$

and

$$T_{ij} = -\frac{e^2 t_c}{m}\int V_i \frac{\partial \varrho_0}{\partial V_j}\,\mathrm{d}v\,, \tag{29}$$

respectively, will become in our dissipative framework

$$J_k = \frac{e^2 t_c m_0}{m X_2}\,E_k\,, \tag{30}$$

$$T_{jj} = \frac{e^2 t_c m_0}{m X^2}\,\delta_{ij}\,, \tag{31}$$

where we have assumed a Maxwell distribution.

This result illustrates, in the case of our example, how the parameters λ and μ of the formalism influence the physical quantities of the system. More explicitly, these quantities represent a decrease (increase) of the electric current and of the conductivity tensor with respect to the corresponding quantities of a conservative condition when $Kt_c(\lambda + \mu) > 1$ (< 1). Thus the parameters of the formulation can be used to determine the best fit for physically established values of $\boldsymbol{J}$ and of T_{ij}. Of course, when the parameters are contracted to the values $\lambda = -\mu = 1$, then all the corresponding expressions for conservative conditions are recovered.

Let us also note that our « perturbative approach » seems to be compatible with the considered example of an elementary dissipative plasma, since also for the conservative formulation the solutions corresponding to (24) are calculated for small deviations from the equilibrium configuration.

In a similar way, other equations for a nonrelativistic or a relativistic plasma can be extended to a dissipative condition either according to the above « perturbative approach » or by means of more general procedures.

In any case, when a dissipative condition appears in a region of a plasma and the true Lagrangian L_C and Hamiltonian H_D are conceived in such a way as to represent the variations of the kinetic and potential energies according to (12), then the Liouville equation of the system will be of the form given by expression (20) when λ and μ are independent of q_k and p_k.

Conceivably, further investigations of the above model may be interesting. For instance, one could study the possible connections between the instabilities of a given region of a plasma and the exchange of energy, momentum or angular momentum between the considered region and the rest of the system.