On Quantum Mechanics

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Abstract

We discuss the axiomatic basis of quantum mechanics and show that it is neither general nor consistent, since its axioms are incompatible with each other and moreover it does not incorporate the magnetic quantization as in the cyclotron motion. A general and consistent system of axioms is conjectured which incorporates also the magnetic quantization.

The neccessity of quantum mechanics (QM) was a result of experimental data for the energy spectrum of atoms which could not be explained in accord with the laws of classical mechanics. Nevertheless in the axiomatic basis of QM there are axioms which are not supported by the experimental results, since the whole system of axioms (1) is introduced *only* as a "plausible generalization" of the first axiom [1]:

$$[\hat{P}_i \quad , \quad \hat{Q}_j] = -i\hbar\delta_{ij} \quad , \quad [\hat{P}_i \quad , \quad \hat{P}_j] = 0 \quad , \quad [\hat{Q}_i \quad , \quad \hat{Q}_j] = 0 \tag{1}$$

where $i, j = 1, ..., k$ and $\hat{P}_i$ and $\hat{Q}_i$ are the momentum and position operators of a quantum system with k degrees of freedom. Note as an importent fact about the application of QM that the usual application of the QM to the energy spectrum of atoms needs only to use the first axiom to quantize the Hamilton operator, but it does not need any use of the other two axioms [2]. Thus these two axioms are not involved yet in any quantum theory. Hence, in view of the fact that the energy quantization was the only application of the system (1), therefore the two last axioms remained without application and it was not possible to prove their compatibility with the first axiom within a concrete physical question.

Nevertheless since these axioms, as the first principles of QM, are based as usual on plausible arguments, but not on other quantum axioms or empirical results. Therefore one can not exclude inconsistencies within the system (1) a periori, so that a general revision of the axiomatic structure of QM seems to be neccessary. Thus the appearance of new quantum effects like the magnetic quantization in the cyclotron motion enforces such a revision, specially if one will describe them in accord with QM. In other words, among others, a reason to revise the axiomatic basis of QM is that the quantum commutator in the two dimensional cyclotron motion [3]:

$$eB[\hat{Q}_m \quad , \quad \hat{Q}_m] = -i\hbar\epsilon_{mn} \quad ; m, n = 1, 2 \quad , \; \epsilon_{mn} = -\epsilon_{nm} = -1 \quad , \tag{2}$$

with Q_m as the relative coordinates of an electron moving in a constant magnetic field B, is not compatible with the system (1).

From dimensional analysis it is obvious that since the action ($\sim \hbar$) has the dimension of momentum times position, i. e.: $P \cdot Q = L^{-1} \cdot L = L^0$ in geometric units. Hence the dimensionality is saved in this case, in view of $dimB \sim L^{-2}$. Therefore also this commutator should define, in principle, a quantum postulate. But if one considers the commutator (2) as a quantum postulate, then the system (1) can be considered neither as the general basis for quantum postulates nor as a consistent algebra.

A more conceptual reason to neglect the cyclotron commutator, as a possible quantum postulate, was the common but a periori belive that

"particles" and "fields" should have two fundamentaly different nature and so they should obey two different type of mechanics or dynamics. Therefore the quantum commutator postulates of particles and fields should be different and without any common relation. Even after the rise of quantum field theory, where quantized fields appear as particles and particles appear as quantum fields, the main difference still remained. Although for example the quantum electrodynamics (QED) is based on the equivalence of the quantum behaviour of electrons and electromagnetic potential field [4].

In accord with QED, in the same manner that quantum mechanical properties of the charged test body ($\sim$ electron) prevent an exact measurment of the electromagnetic field, the quantum electrodynamical properties of the electromagnetic field prevent an exact measurment of the position of the charged test body [4]. In other words the uncertainties of electron causes the uncertainties of the electromagnetic field and vice versa. Thus, in accord with QM where the quantum character of a particle is manifested for example by its uncertainty relations, the quantum character of electron depends on the existence of the uncertainty relations of the electromagnetic field, which manifest the measuring interaction between the electron and the field [4]. Hence QED (of electromagnetic fields) and the QM (of electrons) are two inseparable part of a quantum theory (of particles and fields) and neither is consistent without the other [4].

Nevertheless, it was the mentioned artificial difference between fields and particles with its further consequences which prevented to interpret the phenomenologically introduced commutator in the cyclotron motion, i. e.: $eB[\hat{Q}_i \quad , \quad \hat{Q}_j] = -i\hbar$, as what it is, namely as a quantum commutator postulate: Since in this case the position operators of electron and the magnetic field B appear together in one and the same relation.

The internal incompatibility of the algebra (1) is based, besides of this fact, on the disconnectedness of its axioms, since they are assumed just as a "plausible generalization" [1] partly as the quantum property and partly as the classical properties of a free system: Thus by the last two axioms, the quantum behaviour of, e. g. a quantum particle, is considered to be the same as the classical properties of a classical particle in a free motion. Thus even the obvious relations between the momentum and the position coordinates of a classical particle in a bounded motion are ignored, where for example $P_m = M\dot{Q}_m = \epsilon_{mn} M Q_n \dot{\alpha}$ for $Q_1 = r cos\alpha, Q_2 = r sin\alpha$. In *this case* the Poisson brackets:

$\{P_1 \ , \ P_2\} \propto \{P_1 \ , \ Q_1\}$ and $\{Q_1 \ , \ Q_2\} \propto \{Q_1 \ , \ P_1\}$ are non-trivial [5]. In other words, in the case of bounded motion, the non-triviality of the first bracket of Poisson algebra requires also the non-trivialities of the second and the third brackets. Thus in view of the mentioned correspondance between Poisson- and the Heisenberg algebra, the same requirement should hold also for the corresponding commutators in the system (1) for the related quantum case.

To prove the general contradiction between the axioms of the system (1) let us consider the most minimal case where the algebra (1) can be proved, namely for $i, j = m, n = 1, 2$. We will prove that the first axiom of the system (1) is incompatible with the rest two axioms: This is obvious implicitely from the well known relation between the phase space variables and their quantum operators, in accord to which if two such operators commute with each other, then one of these variables is a function of the other one [6]. On the other hand if two such variables are independent of each other, then the commutator of their operators need not to vanish, which is obvious from the first axiom of the system (1). These relations follows from the mentioned correspondance btween the commutators and Poisson brackets. Thus the asumption of the first commutator in (1), in accord to which $\hat{P}_m$ commute with $\epsilon_{mn}\hat{Q}_n$, but not with $\hat{Q}_m$, means that P_m is a function of $\epsilon_{mn}Q_n$, but not of Q_m, i. e.: $P_m = f(\epsilon_{mn}Q_n)$ and also $P_m \neq f(Q_m)$. Hence in view of $Q_m \neq f(\epsilon_{mn}Q_n)$, it follows that $P_m \neq f(P_n)$ and therefore the related operators need not to commute, i. e.: $[\hat{P}_m \ , \ \hat{P}_n] \neq 0$ [7]. Note that these arguments about the relation between P_m, Q_m and their operators is not in contradiction with the Hamitonian case where the Hamiltonian H is a quadratic function of P_m and Q_m variables, but its operator $\hat{H}$ does not commute with those of $\hat{P}_m$ and $\hat{Q}_m$. Since, not only that in the above discussed case P_m are only linear functions of $\epsilon_{mn}Q_n$, whereas the Hamiltonian is quadratic function of them, but also the direction of conclusion does not contradict the Hamiltonian case. A contradiction with the Hamiltonian case would appear, if we required the oposite direction of conclusion (see [7]) [8].

To prove this fact explicitely let us consider the wave function of the system (1), with respect to which the commutators of (1) can be proved directly, to be in the position representation, i. e.: $\Psi((1)) :=$

$\Psi(Q_1, Q_2)$. Hence the momentum operators should act as differential operators, i. e.: $\hat{P}_m \Psi(Q_1, Q_2) = -i\hbar\partial_m \Psi(Q_1, Q_2) = P_m \Psi(Q_1, Q_2)$ and the position operators should act by multiplication, i. e.: $\hat{Q}_m \Psi(Q_1, Q_2) = Q_m \Psi(Q_1, Q_2)$. On the one hand the assumption of the first postulate in (1), i. e.: $[\hat{P}_1 \ , \ \hat{Q}_1] \neq 0$, $[\hat{P}_2 \ , \ \hat{Q}_2] \neq 0$, $[\hat{P}_1 \ , \ \hat{Q}_2] = 0$ and $[\hat{P}_2 \ , \ \hat{Q}_1] = 0$ demands that, as we discussed above, the momentum variables P_m can not be functions of position variables Q_m but they must be functions of $\epsilon_{mn} Q_n$ variables, i. e.: $P_m = f(Q_n \epsilon_{mn})$. A dependence between momentum- and position variables which is similar to the above introduced example of bounded motion. On the other hand, if so then the second commutator of the standard quantum algebra (1) is not more fulfilled in this case, since as in the bounded motion, this commutator is not trivial for the case of a linear dependence: $P_m \propto \epsilon_{mn} Q_n$. Thus $[\hat{P}_1 \ , \ \hat{P}_2]\Psi(Q_1, Q_2) \neq 0$ by calculation. Hence the contradiction between the first and the second commutators in the algebra (1) can be proved also explicitely: Thus in the quantized bounded motion which is similar to the cyclotron motion of electron in a magnetic field, the second commutator is given by: $[\hat{P}_1 \ , \ \hat{P}_2]\Psi(Q_1, Q_2) = \hat{P}_1(P_2 \cdot \Psi) - \hat{P}_2(P_1 \cdot \Psi) = -2i\hbar M \dot{\alpha} \Psi(Q_1, Q_2)$ in contrast to the system (1), since $P_m = \epsilon_{mn} M \cdot Q_n \cdot \dot{\alpha}$ and $\hat{P}_n = -i\hbar\partial_n$. Therefore in view of the fact that in the case of electromagnetic interaction the dimensionless electron charge e should be involved as a coupling constant, one may set for the cyclotron motion: $\dot{\alpha} = \omega_c = \dfrac{eB}{2M_e}$. So that for the electron coupled to the electromagnetic field B, the commutator: $[\hat{P}_1 \ , \ \hat{P}_2]$ results in:

$$[\hat{P}_1 \ , \ \hat{P}_2] = i\hbar e B \ , \tag{3}$$

which is in contradiction of with the second axiom in (1).

One can even prove that the third commutator in the system (1) results, for the same quantized bounded motion where the position operators in the momentum representation are given by: $\hat{Q}_m = i\hbar\partial_{P_m}$, in: $[\hat{Q}_m \ , \ \hat{Q}_n]\Psi(P_1, P_2) = -2i\hbar(M\dot{\alpha})^{-1}\Psi(P_1, P_2)$. Thus it can be rewritten by the commutator (2) for the case of cyclotron motion, again in accord with $\dot{\alpha} = \omega_c = \dfrac{eB}{2M_e}$.

Another conceptual basis to chooose the algebra (1) was also the a periori concept of "free" quantum particle, e. g. an "electron" without interaction with any field, thus such a free particle have commuting position- and also momentum operators. Nevertheless as it is known from QED [4], such a "free" electron does not exists within the context of QED, since as it is discussed above an "electron" without interaction with quantized electromagnetic field can not be considered as a quantum particle: Thus, in accord with Heisenberg's argument, in order that the uncertainty relations $\Delta P_i \cdot \Delta Q_i \geq \hbar$ are given for an electron as a quantum particle, there must be given an uncertainty relations for the measurment of electron by an electromagnetic field in accord with: $\Delta G_i \cdot \Delta Q_i \geq \hbar$, where G_i is the field momentum of the observing electromagnetic field [4]. In other words the measurment or interaction of an electron with the electromagnetic field, which is manifested by the last uncertainty relation, is the presupposition for quantum character of electron which is manifested by the uncertainty relation $\Delta P_i \cdot \Delta Q_i \geq \hbar$. Hence the existence of uncertainty relation $\Delta P_i \cdot \Delta Q_i \geq \hbar$ depends on the existence of the uncertainty relations $\Delta G_i \cdot \Delta Q_i \geq \hbar$ which manifests the interaction between the electron and the electromagnetic field. Therefore in view of the QM fact that the existence of uncertainty relations is equivalent to the existence of related commutators, the discussed interaction between electron and the electromagnetic field is the presupposition for the correctness of the first commutator in (1). Hence a system of axioms which contains the first commutator in (1), can not apply to a "free" electron, but it applies to a particle with electromagnetic interaction. As a first consequence, the system (1) which presuppsoes the existence of free quantum particle is inconsistent, as we showed above implicitely and explicitely. Moreover in the absence of such a "free" motion, it is no neccessity to assume the second and the third commutators in (1) which manifest the free motion [9], but one should postulate other axioms which are suitable for a bounded motion. Nevertheless we will show that the electron as a quantum particle, not only in the cyclotron motion, but in view of its general neccessary interaction with the electromagnetic field which manifests the quantum character of electron, does not obey the system of axioms (1), but it should obey a system of axioms with non-trivial second and third commutators. This system will be the general and consistent one for a quantum particle like electron, since despite of the system (1) it considers the neccessary coupling of electron, as a quantum particle, to the electromagnetic field. Therefore it will be also the quantum algebra of quantum electrodynamical effects of electron, like the cyclotron motion and the flux quantization.

To prove this, first note that the field momentum of electromagnetic field: $G_i = \int \epsilon_{ijk} E_j B_k d^3x$; $i, j, k = 1, 2, 3$ is equal to eA_i where A_i is the electromagnetic potential. This equality can be derived for E_j and B_k as the solutions of the inhomogeneous Maxwell equations for an electromagnetic field coupled to a single electron. If one uses the Gauss' law for E_j and the integral $A_i = \int \epsilon_{ijk} B_k dx_j = \epsilon_{ijk} B_k x_j$, in view of $divB = 0$. Hence the above introduced uncertaity relation $\Delta G_i \cdot \Delta Q_i \geq \hbar$ can be rewritten by $e\Delta A_i \cdot \Delta Q_i \geq \hbar$ which should be considered as the presupposition for the quantum character of electron.

The argument to prove the general neccessity of non-trivial commutators for a quantum particle like elektron, is based on the fact that on the one hand the quantized electromagnetic potential, the photon, possess two degrees of freedom or two components A_m which are given in two dimensions by $A_m = B \cdot Q^n \epsilon_{mn}$ [10]. On the other hand in accord with the above analysis the quantum character of electron which is manifested by its uncertainty relations, presupposes the uncertainty relations $e\Delta A_i \cdot \Delta Q_i \geq \hbar$. Therefore in view of the fact that such an interaction, to determine the position of electron in the Q_1- direction, causes also an uncertainty ΔQ_2 in the position of electron in the Q_2-direction, in accord with: $\Delta A_1 = B \cdot \Delta Q_2$ and $e\Delta A_1 \Delta Q_1 = eB\Delta Q_2 \Delta Q_1 \geq \hbar$. Hence in view of the QM equivalence between commutators and related uncertainty relations, the existence of the uncertainty relation $eB\Delta Q_1 \Delta Q_2 \geq \hbar$ for an electromagnetically measured electron is equivalent to the existence of the commutor (2), i. e. $eB[\hat{Q}_i \quad , \quad \hat{Q}_j] = -i\hbar$, for the electron as a quantum particle. Then this commutator should replace the third commutator in (1). Thus in accord with this replacement and the above analysis of the measuring interaction between electron and electromagnetic field which results in commutator (3), also the second commutator in (1) should be replaced by the commutator (3).
In other words the new general and consistent system of axioms are given by: $(i, j = m, n)$

$$[\hat{P}_m \quad , \quad \hat{Q}_n] = -i\hbar\delta_{mn} \quad , \quad [\hat{P}_m \quad , \quad \hat{P}_n] = i\epsilon_{mn}\hbar eB \quad , \quad [\hat{Q}_m \quad , \quad \hat{Q}_n] = -i\epsilon_{mn}\hbar(eB)^{-1} \tag{4}$$

To see the consistency of this system of axioms, note that considering the quantization condition $P_m = eA_m$ which is used also in the flux quantization [11], these tree commutators are equivalent to each other by: $P_m = eA_m = eB \cdot Q^n \epsilon_{mn}$. In other words one can consider the algebra (4) as various representations of one and the same commutator:

$$[\hat{Q}_m \quad , \quad \hat{Q}_n] = -i\epsilon_{mn}\hbar(eB)^{-1} \quad , \quad B \cdot \hat{Q}_m = \hat{P}_n \epsilon_{nm} \tag{5}$$

In conclusion let us denote that the classical limit of cyclotron motion, i. e. the $B \to 0$ limit is equivalent to the classsical limit: $\hbar \to 0$ where the area and the radious of motion surface become very large, i. e. close to the rectilinear motion which can be considered as a bounded motion with an infinite large radious and area. Moreover note that the algebra (5), i. e.: $eB\epsilon_{nm}[\hat{Q}_m \quad , \quad \hat{Q}_n] = -i\hbar$, describes beyond the cyclotron motion also the flux quantization which is given usually by $e\int\int F_{mn} dQ^m \wedge dQ^n = e\epsilon_{mn}B\int\int dQ^m \wedge dQ^n = e\oint A_m dQ^m = Nh \quad , N \in \mathbf{Z}$ for a constant magnetic field B. Since in view of $e\epsilon_{mn}B\int\int dQ^m \wedge dQ^n = e\oint \epsilon_{mn}B \cdot Q^m \wedge dQ^n$ and in accord with the QM equivalence between the two quantization postulates in the canonical quantization, i. e. $\int\int dP_m \wedge Q^m = \oint P_m dQ^m = Nh$ and $[\hat{P}_m \quad , \quad \hat{Q}_n] = -i\hbar\delta_{mn}$, the integral form of flux quantization relation is equivalent to the quantum commutator axiom: $eB\epsilon_{mn}[\hat{Q}_m \quad , \quad \hat{Q}_n] = -i\hbar$. Thus a comparison between the mentioned canonical quantization integrals and the flux quantization integrals manifests beyond the flux quantization requirement [11] also the neccesity of relation $P_m = eA_m = eB \cdot Q^n \epsilon_{mn}$.

Footnotes and references

References

[1] Note that the system (1) is introduced as the "plausible generalization" of the first axiom: $[\hat{P}_i \quad , \quad \hat{Q}_j] = -i\hbar\delta_{ij}$ which is the main axiom of the system (1). See: M. Born, W. Heisenberg, P. Jordan, Z. Phys. 35, 557 (1926).

[2] In other words there are no terms of the: $[\hat{P}_i \quad , \quad \hat{P}_j]$ or $[\hat{Q}_i \quad , \quad \hat{Q}_j]$ form in the energy quantization.

[3] H. Aoki: Rep. Prog. Phys. 50, (1987), 655. Note that also in the Landau model of magnetic quantization only the first axiom in (1) is used to quantize the energy, since for this one needs only the first axiom. Thus the cyclotron commutator is not considered there as a canonical commutator postulate in the sense that we do, but it was considered only as a phenomenological relation.

[4] W. Heitler: The Quantum Theory of Radiation, Third Edition, (Dover Publications, Inc. New York 1984), chapters (II-1, II-7 and II-9). Specially note the following remark in (II-1) which is based on the Heisenberg's argument: " In order that $\Delta P \cdot \Delta Q \geq \hbar C$ should hold ..., it is necessary that there should be for the *light beam* an *uncertainty relation* similar to $\Delta P \cdot \Delta Q \geq \hbar C$ ". Then the author

presents for the desired uncertainty relation: $\Delta G_x \cdot \Delta x \geq \hbar C$ where G is the field momentum of the electromagnetic field.

[5] Recall that the quantum Heisenberg algebra (1) corresponds to the classical Poisson algebra $\{P_i \ , \ Q_j\} = \delta_{ij} \ , \ \{P_i \ , \ P_j\} = \{Q_i \ , \ Q_j\} = 0$. Note that the previous dimensional considerations are valid also here.

[6] In other words the conclusion is: $[\hat{A} \ , \ \hat{B}] = 0 \ \rightarrow A = f(B)$. Note that the oposite conclusion, i. e.: $C = f(D) \not\rightarrow [\hat{C} \ , \ \hat{D}] = 0$ is not intended here. In view of the fact that the axioms of QM are given as commutators of operators, therefore one needs here only the first conclusion.

[7] In other words the direction of conclusion is: $P_m \neq f(P_n) \rightarrow [\hat{P}_m \ , \ \hat{P}_n] \neq 0$. The oposite direction will be: $[\hat{A} \ , \ \hat{B}] \neq 0 \rightarrow A \neq f(B)$, which is not intended here.

[8] Nevertheless note also that on a two dimensional manifold, in view of the fact that the area ($\sim L^2$) is an invariant, therefore on such a manifold $L^2 \sim L^0 \sim (constant)$. This means that in the *two dimensional* case under consideration: $(\sim Q_m; m = 1, 2)$ where also $P_m = f(\epsilon_{mn} Q_n)$, the quadrats of Q_m and P_m are constants of the motion and a quadratic dependence of Hamiltonian H of these variables, is equivalent to a dependence of constants only.

[9] The free motion or the motion of a free particle is the motion of a particle that stands under no potential or no force or no field strength. In other words for a free motion one has $B = 0$ everywhere.

[10] The A_m component of the homogeneous magnetic field B on two dimensional manifolds which is the case here, can be given, up to a constant potential, by $A_m = \epsilon_{mn} B \cdot x^n$. Since then the two form $B \sim F = dA \oplus Harm^1$ on a *two dimensional* manifold is divergenceless and gradientless, as expected. The term $Harm^1$ represents the harmonic one forms.

[11] Note that the flux quantization requirement of vanishing of current density in the *contour* region is equivalent to the vanishing of the velocity of electrons in this region which is also equivalent to $P_m = eA_m$ in this region, since: $V_{(contour)} = \dfrac{1}{m_e}(P_m - eA_m)_{(contour)} = 0$.